Shear flow instabilities and Rossby waves in barotropic flow in a rotating annulus

T. H. Solomon, W. J. Holloway, and Harry L. Swinney
Center for Nonlinear Dynamics and Department of Physics, University of Texas at Austin, Austin, Texas 78712

(Received 23 July 1992; accepted 6 April 1993)

The primary instability of an azimuthal jet is studied experimentally in a rotating annulus with a rigid upper lid (infinite Rossby deformation radius) and a sloping bottom (topographical beta effect). An azimuthal jet is produced by the action of the Coriolis force on fluid pumped between concentric rings of sources and sinks in the bottom of the annulus. The flow is essentially two dimensional by the Taylor–Proudman theorem. Velocity measurements are made with hot-film probes and particle streak photography. For small forcing flux $F$, the jet is axisymmetric and has a $1/r$ velocity profile bounded by narrow free shear layers on each side. At a critical value of $F$, the inner shear layer becomes unstable to the formation of a propagating chain of vortices; at a larger value of $F$ the outer shear layer also becomes unstable. The critical values of the mode numbers, wave speeds, and $F$ at different annulus rotation rates are in good accord with a linear stability analysis by Lee and Marcus. At onset of instability the vortex chains are similar to those formed by a Kelvin–Helmholtz instability of a free shear layer, but this flow also has some properties typical of a Rossby wave flow—e.g., corotating jets have vortex chains with smaller wave speeds than counter-rotating jets. This asymmetry is enhanced as $F$ is increased beyond instability onset, since growth in the size of the vortices results in a decrease of Rossby number. At sufficiently large $F$ the vortex chains lock, resulting in a broad wavy jet with the same number of vortices on both sides. For corotating jets these states have significantly smaller wave speeds than the unlocked state, and this small speed is in accord with that predicted for a sinuous Rossby wave. However, the counter-rotating jets do not satisfy this relation but rather continue to have properties similar to Kelvin–Helmholtz waves.

I. INTRODUCTION

Laboratory and theoretical studies of fluid flows in well-controlled rapidly rotating systems are of particular interest because of similarities to atmospheric and oceanic flows and to plasmas in the presence of a strong uniform magnetic field. The present work concerns an incompressible flow with mechanical forcing. The two-dimensionality of this barotropic flow permits precise comparison between experiment and theory, even well beyond the onset of instability where in both experiments and simulations it would be difficult to resolve fully a three-dimensional velocity field produced by thermal forcing.

Barotropic flows with sufficiently strong shear layers are unstable to the formation of vortex chains, similar to those formed by the Kelvin–Helmholtz instability of a free shear layer. Additional mechanisms for instability are present if there are variations in the topography of the tank bottom (beta effect), analogous to variations in the Coriolis force with latitude in geophysical flows or to variations in ion density in plasmas. A restoring force parallel to the topography gradient results in Rossby waves, which are analogous to drift waves in plasmas. Rossby waves have been studied extensively in the more complicated baroclinic (density-driven) flows, which are inherently three dimensional. Various experiments have also studied, at large Reynolds number, barotropic flows with a beta effect, where Rossby effects are important. Surprisingly, less is known about the basic instability leading to the formation of Rossby waves. Manin and Chernous'ko studied the basic instability of a narrow barotropic jet, but their work does not distinguish between shear-induced and beta effects.

In this article we extend the previous work to examine situations where mechanisms for both shear and Rossby wave instabilities are present, as is frequently the case in geophysical and plasma flows. The focus here is on the primary instability of an axisymmetric jet and on the properties of the resulting wave states as the forcing parameters are increased beyond onset. Well above onset, interactions between two spatially separated vortex chains complicate the time dependence. With sufficiently strong forcing, the chains lock, resulting in a simpler state that for corotating flows has properties characteristic of sinuous Rossby waves.

Section II defines the problem and discusses previous work. Section III describes the apparatus and measurement techniques. Section IV presents the experimental results and compares them with theory. Section V is a discussion.

II. BACKGROUND AND PREVIOUS WORK

A. Dimensionless parameters and two-dimensionality

Given relevant length and velocity scales $L$ and $U$, a rotating incompressible fluid can be described by three dimensionless parameters: (1) the Reynolds number $R = UL/v$ (where $v$ is the kinematic viscosity), (2) the
Rossby number \( e = U/2\Omega L \) (where \( \Omega \) is the angular frequency of the system), and (3) the Ekman number \( E = v/\Omega L^2 \). The relevant length and velocity scales for the current experiments will be discussed in Sec. IV B. Barotropic flows in rapidly rotating systems (large \( \Omega \)) with vanishing \( \varepsilon \) and \( E \) (geostrophic flows) are two dimensional by the Taylor–Proudman theorem.\(^6\)

**B. Shear flow instabilities**

In the absence of rotation, a free shear layer is always unstable to the formation of a vortex chain with small wave number.\(^25\) The problem is modified if the system is rotating. The stability of a free shear layer in a rotating flow was examined initially by Hide and Titman\(^6\) and by Busse,\(^9\) and, more recently, by Niino and Misawa.\(^7\) Such a layer is unstable to the formation of vortices only if the velocity gradient in the shear layer exceeds a critical value. Hide and Titman measured neutral stability curves for the instability as a function of \( E \) and \( \varepsilon \), and determined mode numbers for the resulting vortex chains. Niino and Misawa defined the Reynolds number in terms of the width of the shear layer and the velocity drop across the shear layer, and they found that the Reynolds number for instability was independent of \( E \) and \( \varepsilon \). There was no beta effect in the studies of Hide and Titman, Busse, or Niino and Misawa.

The vortex chain that results from a shear instability in a rotating fluid without a beta effect is similar to that formed by the Kelvin–Helmholtz instability.\(^8\) If the velocity profile is antisymmetric with respect to the average velocity of the shear layer, the propagation speed \( c \) of the chain is this average velocity, independent of the wave number. The streamfunction for this dispersionless vortex chain, when viewed from a reference frame moving with the vortices, is the familiar “cat’s eye” pattern first described by Kelvin\(^8\) [Fig. 1(a)].

**C. Rossby waves**

The vorticity of a fluid element changes when the depth of a rotating flow changes.\(^2\) This effect is analogous to the beta effect—the change in vorticity experienced by a fluid element moving in a spherical shell, where the Coriolis force changes with latitude. The resultant restoring force is responsible for the formation of Rossby waves.

The linear stability of small amplitude disturbances in a rotating flow with a beta effect can be analyzed by considering the Rayleigh–Kuo equation in the slab approximation (i.e., neglecting curvature),

\[
(U_0 - c) \left( \frac{d^2\phi}{dy^2} - k^2 \phi \right) + \beta \frac{d^2U_0}{dy^2} \phi = 0, \quad (1)
\]

where \( y \) is the cross-jet coordinate, \( \beta \) is the magnitude of the beta effect (defined for this system in Sec. III A), \( U_0(y) \) is the velocity profile of the jet, and the streamfunction of the perturbation is \( \psi(x,y,t) = \phi(y) \exp[-i k(x - ct)].\)

Previous experiments\(^19\) indicate that the velocity profile for a jet with \( \beta > 0 \) (a corotating jet in this experiment) with strong source-sink forcing is typically \( U \operatorname{sech}^2(\eta \gamma) \), where the constant \( U \) is the maximum jet velocity. The linear stability of this jet profile (called the Bickley jet) has been analyzed extensively in previous studies.\(^19,22,26\) Three neutral modes have been found for this jet: a sinuous mode, a varicose mode, and a singular (dispersionless) neutral mode. For the sinuous mode, \( \phi(y) \) as well as \( U_0(y) \) has a \( \operatorname{sech}^2(\eta \gamma) \) dependence. The neutral stability curve for this mode is given by

\[
\beta = \frac{k^2 U}{6} \left( 4 - \frac{k^2}{\beta} \right). \quad (2)
\]

For \( 0 < \beta < 2U_k^2/3 \), there are two neutrally stable sinuous modes, a “fast” mode and a “slow” mode\(^22\) with wave speeds\(^19\)

\[
c_{1,2} = \frac{U}{3} \left( 1 \pm \sqrt{1 - \frac{3\beta}{2U_k^2}} \right), \quad (3)
\]

and dispersion relations\(^19,26\)

\[
c_{1,2} = \frac{U}{6\beta^2} k_{1,2}^2, \quad (4)
\]

where the subscript 1 (2) refers to the fast (slow) mode. The wave speeds can be expressed in a simpler form by combining Eqs. (2) and (4) to eliminate the \( l \) dependence:\(^22,28\)

\[
c = \frac{3}{2}U - \beta/k^2. \quad (5)
\]

It should be noted that this relation (which applies equally to both the slow and fast modes) is not a dispersion relation since the width of the jet is not held constant.\(^22\)

Although derived for a particular jet profile, Eq. (5) is in a form that clarifies the importance of the beta effect on the phase speeds for sinuous Rossby waves propagating on a wide variety of jets.\(^29\) For \( \beta = 0 \), the waves are dispersionless with phase speeds equal to the characteristic jet velocity \( 2U/3 \), similar to the dispersionless vortex chains discussed in the previous section. If \( \beta \) is positive (as is the case...
for atmospheric flows), Rossby waves propagate opposite to the direction of rotation, relative to the characteristic zonal velocity. As a result, Rossby waves that propagate on corotating jets have smaller frequencies and wave speeds than those that propagate on counter-rotating jets. This east–west asymmetry is a distinct signature of Rossby waves.

A typical streamfunction for a Rossby wave is shown in Fig. 1(b), viewed from the reference frame of the wave. Unlike the vortex chains resulting from instability of a free shear layer, the islands of vorticity alternate in sign, with a wavy jet separating the islands.

Several previous studies examined Rossby waves in three-dimensional baroclinic flows driven by imposed temperature gradients, Rosby waves have also been studied in barotropic systems with moving sources and sinks, but these studies did not measure neutral stability curves for the instabilities. No previous experimental study has characterized the instability of an axisymmetric jet in a barotropic flow with a beta effect or compared the results with linear stability theory.

D. Linear stability analysis

Lee and Marcus have solved the two-dimensional quasigeostrophic shallow-water equations for our geometry and boundary conditions. Their results will be published in detail elsewhere, but we will briefly describe the analysis since we will compare some of their predictions with our observations. Lee and Marcus assume an infinite Rossby deformation radius (since the experiments have a rigid lid) and include dissipation from both molecular viscosity and Ekman friction; no slip boundary conditions are used at the side boundaries. The boundary layers at the side walls are fully resolved so no assumptions about these boundary layers are needed. The circular rings of forcing holes in the bottom of the tank (see the next section) are modeled as axisymmetric slits (sources and sinks) of fixed width. The equations are solved in two different ways. One technique solves the steady-state nonlinear ordinary differential equations to obtain the axisymmetric basic flow, and then a linear stability analysis is performed on this numerically determined basic flow to determine the onset of instability. The other technique is an initial value method, initialized with a linear combination of the unstable axisymmetric solution and one or more eigenmodes with arbitrary amplitude and phase (the results are insensitive to the initialization).

The two analysis techniques yield results in good accord with each other. The primary instability has been analyzed for both $\beta=0$ and for positive $\beta$ (as in our experiments). The order in which the different eigenmodes become unstable is found to be somewhat dependent on the width assumed for the source/sink rings; this width is the only adjustable parameter. The theoretical results given in our figures were obtained for rings with an approximately Gaussian profile with full width at half-maximum of 1.2 cm (chosen as an approximation of the width of the shear layer).

III. EXPERIMENTAL APPARATUS AND TECHNIQUES

A. Description of annulus

The annulus used in these experiments is shown in Fig. 2 and the parameter values are given in Table I. The working fluid is water (kinematic viscosity $v=0.0094 \text{ cm}^2/\text{sec}$ at $23 ^\circ\text{C}$) or water–glycerol mixtures ($v=0.030, 0.033, \text{ or } 0.100 \text{ cm}^2/\text{sec}$). The fluid is confined between two cylinders of radii $r_1 = 10.8 \text{ cm}$ and $r_2 = 43.2 \text{ cm}$. The outer cylinder is made of Plexiglas (to allow optical access) with three external steel bands for reinforcement. The tank has a rigid Plexiglas upper lid, and the bottom has a slope. The beta coefficient is given by $\beta = 2\Omega h/r$, where $h$ is the mean depth of the fluid. Slight warping in the upper lid due to centrifugal forces results in an error in the fluid depth of up to about 0.2 cm.

The annulus rotates rigidly about its axis. Electrical connections to equipment and probes on the rotating platform are made via 16 slip rings and brushes. Rotation frequencies $\Omega/2\pi$ range from 0 to 2.5 Hz. The rotation rate is measured with a rotary encoder that is mounted in the tank.

<table>
<thead>
<tr>
<th>Table I. System parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner radius $r_1$</td>
</tr>
<tr>
<td>Outer radius $r_2$</td>
</tr>
<tr>
<td>Depth $h(r_1)$</td>
</tr>
<tr>
<td>Depth $h(r_2)$</td>
</tr>
<tr>
<td>Bottom slope</td>
</tr>
<tr>
<td>Radial of port rings:</td>
</tr>
<tr>
<td>inner</td>
</tr>
<tr>
<td>center</td>
</tr>
<tr>
<td>outer</td>
</tr>
<tr>
<td>Diameter of port ring holes</td>
</tr>
<tr>
<td>(120 each ring)</td>
</tr>
<tr>
<td>Wide forcing configuration</td>
</tr>
<tr>
<td>inner (18.9 cm) and</td>
</tr>
<tr>
<td>outer (35.1 cm) ports</td>
</tr>
<tr>
<td>Narrow forcing configuration</td>
</tr>
<tr>
<td>center (27.0 cm) and</td>
</tr>
<tr>
<td>outer (35.1 cm) ports</td>
</tr>
<tr>
<td>Maximum flow rate $F$ of pump</td>
</tr>
</tbody>
</table>
coaxially on the shaft of the annulus. Small fluctuations in the rotation frequency (with amplitude up to 0.01 Hz) arise from imperfections in the belt drive and from feedback oscillations in the controlling mechanism, but these fluctuations do not affect significantly the dynamics since the time scale for the oscillations (~1 sec) is much smaller than the typical Ekman spin-down time (~15 sec).

A positive displacement gear pump mounted inside the inner cylinder pumps fluid between concentric rings of holes (forcing rings) in the bottom of the tank. The azimuthal jet velocity is typically two orders of magnitude greater than the forcing flow. There are three forcing rings (see Table I), each with 120 equally spaced holes divided into six independent sectors; in contrast, earlier experiments with this annulus had concentric rings with only six holes each.\textsuperscript{14,15,37} The pressure drop along each ring is much smaller than the drop across the holes (the maximum ratio is 0.2), ensuring spatially uniform pumping.\textsuperscript{38}

In the present work, one ring is an inlet, another is an outlet, and the third is closed. Two configurations are used: "wide" forcing, with the inner and outer rings active, and "narrow" forcing, with the center and outer rings active. Thermal convection could obscure measurements of the onset of barotropic instabilities. To minimize this effect and to maintain a constant kinematic viscosity $\nu$, temperature stability is maintained with the room's air conditioning system and fans that circulate air near the tank. Temperature drifts are less than 0.5 °C during an experimental run lasting up to 6 h, limiting variations in $\nu$ to 1%. Spatial nonuniformities in the temperature are minimized before starting an experiment by repeatedly cycling the rotation rate from 0 to 1.5 Hz, while pumping fluid at 40 cm$^3$/sec for 45 min. The absence of baroclinic instabilities caused by thermal convection in the rotating annulus is then confirmed if the fluid velocity is zero in the absence of pumping.

**B. Point measurements of velocity**

Velocities are measured with single component platinum hot-film probes (TSI, Inc., #1210-60W) that are positioned as shown in Fig. 3(a). Two probes are mounted 10° apart above each of the three forcing rings. A seventh probe is mounted in the bottom, directly below one of the middle ring probes at the top. Mounting the probes directly above the rings provides the greatest sensitivity to the primary instabilities of the shear layers. The mode number and propagation speed of the wave states are determined from the phase shifts between probes at the same radius. The correlation between instabilities in different shear layers is determined using probes in different radii.

Cerasoli\textsuperscript{39} has shown that perturbations from probe mounts as small as 2 mm in diameter can lead to instabilities that are difficult to distinguish from shear layer or Rossby instabilities. We minimize the perturbations by mounting the probes such that only the prongs that support the hot film penetrate 10 mm into the annulus [Fig. 3(b)], well beyond the Ekman layer, whose thickness $\left(\sqrt{\nu/2D}\right)^{1/2}$ is typically 0.3 mm. The mounting holes drilled into the lid are filled at the inside surface with silicone sealant and are flattened with a razor to be level with the surface. The probe prongs are approximately 0.5 mm in diameter at their base, tapering to approximately 0.3 mm near the ends. The Reynolds number for flow past these prongs is typically of order unity. The hot-film probes are operated in constant current mode.\textsuperscript{40} The probes are connected to Wheatstone bridges, which are mounted on the rotating platform to minimize the effects of variations in slip ring resistance. The bridges are driven with a 10 kHz sinusoidal wave with an amplitude of 10 V.

Unless otherwise stated, the hot-film probes are oriented to be sensitive mainly to azimuthal flows (with the hot-film wire pointing radially). The probes are calibrated by changing the rotation rate suddenly to produce a known velocity change.

**C. Flow visualization**

Video and 35 mm still cameras are mounted on a rotating platform above the annulus (Fig. 2). The rotation frequency for the camera platform is independent of the tank rotation frequency so the flow can be visualized either from the reference frame of the rotating tank or from the reference frame of propagating wave structures in the flow. Particle-streak photography is used to characterize the velocity field. Small (~1 mm diam) particles suspended in the fluid follow the flow. Neutrally buoyant particles are made from a combination of carnauba wax and a dust suspension.\textsuperscript{41} A horizontal cross section (approximately 4 cm in height) of the tank is illuminated from the side. The velocity field is deduced from particle streaks in time exposure photographs. The streak coordinates are determined with a digitizing tablet.
FIG. 4. Simultaneous hot-film measurements of the radial component of the velocity at the top and bottom, illustrating two-dimensionality of flow: (a) \( F/F_c = 12 \), (b) \( F/F_c = 35 \). In both plots, the upper (lower) curve corresponds to the signal from the top (bottom) probe (counter-rotating flows, \( v = 0.0094 \text{ cm}^2/\text{sec} \), \( \Omega/2\pi = 2.00 \text{ Hz} \), \( F_c = 1.80 \text{ cm}^3/\text{sec} \)).

IV. RESULTS

A. Two-dimensionality of the flow

The two-dimensional nature of the flow is demonstrated by the data in Fig. 4, which shows time series of hot-film signals measured simultaneously with two probes, one located 1 cm below the top of the tank and the other directly below, 1 cm above the bottom. The two signals are 99% correlated for \( \Omega/2\pi = 2 \text{ Hz} \) and \( F/F_c = 12 \), where \( F_c \) is the total pumping rate at the onset of instability [see Fig. 4(a)]. Even for the chaotic flow in Fig. 4(b), where \( F/F_c = 35 \), the correlation is 97%.

The correlation is shown as a function of \( F/F_c \) in Fig. 5 for \( \Omega/2\pi = 1.0 \) and 2.0 Hz. As expected, the flow is more nearly two dimensional at the higher rotation rate.

Additional evidence for the two-dimensionality is provided by the particle streak photographs. Particles observed with the overhead cameras vary widely in height, yet the trajectories never cross. Dye injected into the tank spreads horizontally, forming a thin sheet. The time scale for the dye to spread in the vertical direction is 15–30 min, which is much longer than the dynamical time scale (~20 sec for a vortex turnover).

B. Axisymmetric state and dimensionless parameters

A streak photograph of a counter-rotating axisymmetric jet below onset of instability is shown in Fig. 6(a). The

FIG. 6. (a) Streak photograph for an axisymmetric counter-rotating jet. The camera is rotating in the reference frame of the tank, and the exposure time is 2 sec. (b) Azimuthally averaged velocity \( v(r) \) from (a). The solid curve is the theoretical prediction, Eq. (6). In the shear layers, uncertainties of about 0.5 cm in position and 0.1 cm/sec in velocity arise from the sparsity of the data, digitization, and the determination of the center position (\( v = 0.033 \text{ cm}^2/\text{sec} \), \( \Omega/2\pi = 2.00 \text{ Hz} \), \( F = 5.9 \text{ cm}^3/\text{sec} \), \( R_i = 10.0 \), \( R_e = 5.38 \)).
jet is confined to the region between the source and sink rings. The velocity profile, obtained from an analysis of digitized streaks, is compared in Fig. 6(b) with a theoretical prediction,\textsuperscript{20,42,43} \textsuperscript{20,42,43}

\begin{equation}
\begin{aligned}
  \nu(r) &= \frac{F}{2\pi} \left( \frac{\Omega}{v} \right)^{1/2} \left( \frac{1}{r} \right), \quad r_i < r < r_o, \\
  \nu(r) &= 0, \quad r < r_i, \quad r > r_o,
\end{aligned}
\end{equation}

where \( r_i \) and \( r_o \) are the radii of the inner and outer rings of source–sink holes. An experimental fit to a function \( A/r \) in the range \( r_i < r < r_o \) gives \( A = 18.2 \pm 0.7 \text{ cm}^2/\text{sec} \), in good agreement with the theoretical prediction of \( A = 18.4 \text{ cm}^2/\text{sec} \). The observed dependence of \( \nu_0 \) on \( F \) is also in good accord with Eq. (6), as Fig. 7 illustrates.

Figure 6(b) shows that there are two free shear layers in the axisymmetric state, one over the ring of source holes and the other over the ring of sinks. If the source and sink rings are far enough apart, these shear layers become unstable independently of each other. It is therefore necessary to define \( R, e, \) and \( E \) independently for the two shear layers, each of which has its own scale \( U \).

We define the Ekman number \( E = \nu/(\Omega h^2) \).\textsuperscript{7} For the velocity scale in \( R \) and \( e \), we use one-half of the velocity drop across the shear layer, as given by Eq. (6). For a length scale we use the width of the shear layer. Our present measurements have insufficient resolution to determine this width accurately; therefore, we approximate the shear layer width by the Stewartson layer thickness\textsuperscript{23}

\begin{equation}
L_s = (E/4)^{1/4} h.
\end{equation}

This approximation neglects broadening effects from the forcing\textsuperscript{20} and the \( E^{1/3} \) layer in which there is a vertical flow (see discussion in Ref. 7). Typically, \( L_s \approx 1 \text{ cm} \), much smaller than the depth (18.7 cm in the center) and the interport spacing (8.1 cm for narrow forcing and 16.2 cm for wide forcing).

The importance of the curvature of the shear layer depends on the aspect ratio \( \Gamma = r_j/L_s \), where \( r_j \) is the radius of the shear layer [with \( j = i \) (inner ring) or \( o \) (outer ring)]. Niino and Misawa\textsuperscript{7} have shown that curvature effects are negligible for \( \Gamma \geq 25 \); for our three forcing rings, \( \Gamma \approx 18, 27, \) and 35.

With these velocity and length scales, \( R \) and \( e \) for each shear layer are given by

\begin{equation}
R_j = \frac{F}{2^{5/2} \pi r_j} \Omega^{1/4} \nu^{-5/4} h^{1/2},
\end{equation}

\begin{equation}
e_j = \frac{F}{2^{5/2} \pi r_j} \Omega^{-1/4} \nu^{-3/4} h^{-1/2}.
\end{equation}

With these definitions \( R \) varies from approximately 5 to 500 and \( e \) varies from approximately 0.05 to 0.2 in these experiments. However, Eqs. (7)–(9) are valid only below and slightly above the onset of instability; broadening of \( Ls \) by instabilities results in \( R \) as large as 3000 and \( e \) as small as 0.01 in these experiments.
C. Primary instability: Shear-induced vortex chains

Since $R_i$ is larger than $R_o$ for any fixed value of the forcing $F$, there is a small range in $F$ where the inner shear layer is unstable while the outer shear layer remains axisymmetric. Figure 8 shows velocity time series and power spectra obtained in this regime with probes in the inner and outer shear layers. Periodic oscillations are clearly visible in the inner shear layer velocity time series [see Fig. 8(a)]. The corresponding spectrum, Fig. 8(b), is singly periodic, corresponding to a traveling chain of eight vortices. The velocity in the outer shear layer, however, is essentially time independent. The corresponding power spectrum, Fig. 8(c), has, in addition to the background noise, only a small peak at 0.106 Hz, which corresponds to a very weak perturbation of the outer shear layer by the vortices at the inner shear layer.

Further increase in $F$ results in destabilization of the outer shear layer, as is illustrated by the velocity time series, power spectra, and correlation functions in Fig. 9. Just beyond the onset of this regime the width of the vortices associated with each shear layer is much smaller than the distance between the forcing rings; hence the vortex chains at the inner and outer shear layers have different dominant frequencies, phase speeds, and mode numbers. The ratio $f^{(i)} / f^{(o)}$ of the dominant frequencies at the inner and outer rings, respectively, apparently varies monotonically with $F$; thus there is a range in $F$ in which the flow is essentially quasiperiodic—there are two dominant incommensurate frequencies $f^{(i)}$ and $f^{(o)}$ plus noise. The independence of the vortex chains at the two shear layers is indicated by the negligible correlation [Fig. 9(d)]. A streak photograph of a velocity field in this regime is shown in Fig. 10. The inner vortex chain with mode number $m_i=7$ is well separated from the outer vortex chain with $m_o=12$. The spatially separated vortex chains of Fig. 10 each have the “cat’s eye” structure typical of shear-induced instabilities [cf. Fig. 1(a)].

Neutral stability curves, mode numbers, and oscillation frequencies at onset are presented in Table II. The experimental results for the inner shear layer are compared in Fig. 11 with the theoretical predictions of Lee and Marcus. There is good agreement between experiment
TABLE II. Experimental values of the parameters at the onset of instability of the axisymmetric jet with the wide forcing configuration.

<table>
<thead>
<tr>
<th>$\Omega/2\pi$ (Hz)</th>
<th>$F_c$ (cm$^3$/sec)</th>
<th>$f$ (Hz)</th>
<th>$m$</th>
<th>$L_c$ (cm)</th>
<th>$E$ ($10^{-5}$)</th>
<th>$R$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>50.6</td>
<td>0.062</td>
<td>7</td>
<td>1.12</td>
<td>6.07</td>
<td>17.0</td>
<td>0.138</td>
</tr>
<tr>
<td>1.00</td>
<td>48.4</td>
<td>0.082</td>
<td>8</td>
<td>1.04</td>
<td>4.55</td>
<td>17.5</td>
<td>0.123</td>
</tr>
<tr>
<td>1.25</td>
<td>46.2</td>
<td>0.084</td>
<td>8</td>
<td>0.98</td>
<td>3.64</td>
<td>17.6</td>
<td>0.111</td>
</tr>
<tr>
<td>1.50</td>
<td>45.0</td>
<td>0.107</td>
<td>9</td>
<td>0.94</td>
<td>3.03</td>
<td>18.0</td>
<td>0.103</td>
</tr>
<tr>
<td>1.75</td>
<td>41.2</td>
<td>0.082</td>
<td>9</td>
<td>0.90</td>
<td>2.60</td>
<td>17.1</td>
<td>0.091</td>
</tr>
<tr>
<td>2.00</td>
<td>41.2</td>
<td>0.107</td>
<td>10</td>
<td>0.87</td>
<td>2.28</td>
<td>17.7</td>
<td>0.088</td>
</tr>
<tr>
<td>0.50</td>
<td>47.4</td>
<td>0.057</td>
<td>7</td>
<td>1.24</td>
<td>9.10</td>
<td>14.4</td>
<td>0.144</td>
</tr>
<tr>
<td>0.75</td>
<td>43.7</td>
<td>0.067</td>
<td>7</td>
<td>1.12</td>
<td>6.07</td>
<td>14.7</td>
<td>0.119</td>
</tr>
<tr>
<td>1.00</td>
<td>39.5</td>
<td>0.080</td>
<td>7</td>
<td>1.04</td>
<td>4.55</td>
<td>14.3</td>
<td>0.101</td>
</tr>
<tr>
<td>1.50</td>
<td>35.7</td>
<td>0.093</td>
<td>7</td>
<td>0.94</td>
<td>3.04</td>
<td>14.2</td>
<td>0.082</td>
</tr>
<tr>
<td>2.00</td>
<td>33.7</td>
<td>0.117</td>
<td>8</td>
<td>0.87</td>
<td>2.28</td>
<td>14.5</td>
<td>0.072</td>
</tr>
<tr>
<td>1.50</td>
<td>8.24</td>
<td>0.064</td>
<td>9</td>
<td>0.70</td>
<td>0.91</td>
<td>14.9</td>
<td>0.047</td>
</tr>
<tr>
<td>1.50</td>
<td>13.8</td>
<td>0.054</td>
<td>14</td>
<td>0.74</td>
<td>0.84</td>
<td>13.7</td>
<td>0.040</td>
</tr>
</tbody>
</table>

and theory. The small differences between the measured and predicted stability curves may be due to the slight warping of the upper lid, which would enhance the beta effect in the vicinity of the inner shear layer. The mode numbers $m$ agree with theory, except for a few measurements where the theoretical and observed values differ by unity. These differences could arise for several reasons: (i) measurements are necessarily made slightly above $F_c$, not at $F_c$; (ii) the differences occur near values of $\Omega$ where the system is particularly sensitive to noise, i.e., where neutral stability curves for different mode numbers cross; (iii) the predicted mode numbers are sensitive to the width of the axisymmetric slits assumed in the analysis (see Sec. II D).

The neutral stability data are plotted as a function of $R$ and $E$ in Fig. 12. In addition to the data from Fig. 11, this plot includes onset data obtained with a different water-glycerol mixture ($v=0.030$ cm$^2$/sec). Figure 12 shows
that $R_c$ is lower for counter-rotating jets than for corotating jets, and also slightly lower for the outer shear layer than for the inner layer.

In the range of $E$ studied, $R_c$ for each flow direction is independent of $E$ within experimental error. This independence of $R_c$ on $E$ was previously noted and discussed by Niino and Misawa, who argued that the lack of $E$ dependence in their data is consistent with the $E^{3/4}$ dependence of Rossby number found in the experiments of Hide and Titman. The independence of $R_c$ on $E$ in our experiments, which have a beta effect in contrast to the earlier experiments, indicates that the primary instability we observe is a shear instability modified only slightly by the beta effect. This interpretation is strengthened by the fact that our $R_c$ values (e.g., 14.3 and 17.5 for counter-rotating and corotating flows, respectively, at $\Omega/2\pi = 0.5$ Hz) are close to that found by Niino and Misawa (11.7) and to the predictions of Lee and Marcus in the absence of beta effects.

D. Wave speeds and length scales above onset

Given the frequency and mode number of a vortex chain, the propagation speed $c$ is determined. The speed of the vortex chain at the inner shear layer is plotted in Fig. 13 as a function of $\epsilon$ for $R$ above (but near) onset. In this plot, $c$ is normalized by $\Delta v_i=0.94v_0(r_i)$, an approximation of the velocity drop across the shear layer. For moderate to large $\epsilon$, $c$ approaches approximately $\frac{1}{2}\Delta v_i$ for both corotating and counter-rotating flows. For small $\epsilon$, however, corotating jets have vortex chains with $c<\frac{1}{2}\Delta v_i$ and counter-rotating jets have vortex chains with $c>\frac{1}{2}\Delta v_i$.

As the forcing is increased beyond onset, the vortex chains grow in amplitude and the width of the shear layers increases, as the velocity profile in Fig. 14 illustrates. Although the core of the jet is still close in magnitude to the $1/r$ profile of Eq. (6), the shear layers have grown in width to several centimeters. In this regime, Eqs. (8) and (9) are no longer an adequate definitions for $R$ and $\epsilon$.

The wave numbers of the vortex chains decrease with increasing forcing, as has been observed in previous experiments on barotropic and baroclinic instabilities. As in the previous studies, the transitions in wave number are clearly hysteretic. Detailed measurements of the secondary wave-number-changing instabilities have not been made because of complications caused by interactions between the inner and outer vortex chains. Experiments are currently in progress in which an axisymmetric barrier in the annulus isolates the two vortex chains, allowing for more precise measurements of the wave-number-changing instabilities.

E. Frequency and mode locking

At large forcing the vortex width becomes comparable to the spacing between the source–sink rings, and the interacting vortex chains lock at the same frequency, mode number, and propagation speed. This transition is illustrated by velocity time series (Fig. 15) and their spectra (Fig. 16). Just below the critical forcing for locking, time series obtained at the center and outer forcing rings [Figs. 15(a) and 15(b)] are complicated and only weakly correlated [solid curve in Fig. 15(e)]. Beyond the onset of locking, the time series [Figs. 15(c) and 15(d)] oscillate 180° out of phase and are almost perfectly correlated [dashed curve in Fig. 15(e)]. The locking transition is hysteretic, as illustrated by the sequence in Fig. 16.

The locking transition is accompanied by a marked reduction in the complexity of the time dependence (Figs. 15 and 16). In addition, for corotating flows—but not for counter-rotating flows—there is a drop in the dominant frequency of 40%–50% (depending on the mode number). We can compare the observed propagation speed $c$ of the wave above the onset of locking with that predicted by Eq. (5), assuming the jet to have a $\text{sech}^2(r)$ velocity profile in the locked regime (as measured by Sommeria et al. for large forcing). The characteristic jet velocity is $V=2U/3$, where $U$ is the peak jet velocity. The resultant wave speed shift ($c-V$), divided by the value predicted for a sinusoidal

![Fig. 13. Wave speed $c$ within 10% of onset of the onset forcing level $F_c$, showing the asymmetry between corotating ($\square$) and counter-rotating ($\triangle$) jets for small $\epsilon$. The wave speed is normalized by the velocity drop $\Delta v$ across the shear layer ($v=0.100$ cm/sec).](image1)

![Fig. 14. Velocity profile $v(r)$ for the streak photograph in Fig. 10, expressed relative to the reference frame of tank. The profile at this forcing, well above onset, is rounded and departs significantly from the axi-symmetric basic flow, which is shown by the solid curve [Eq. (6)] [F=28 cm/sec, $F_c$ (inner) = 8.2 cm/sec, $F_c$ (outer) = 13.8 cm/sec].](image2)
FIG. 15. Transition to locking of the vortex chains, illustrated by center and outer ring velocity time series and their cross-correlations: (a) and (b), unlocked vortex chains, \( F = 20.0 \) cm\(^3\)/sec; (c) and (d), locked vortex chains, \( F = 22.6 \) cm\(^3\)/sec. (e) Cross-correlation \( \chi \) between center and outer ring signals; the solid curve corresponds to the time series (a) and (b) (unlocked state), and the dashed curve corresponds to (c) and (d) (locked state) (corotating jet, \( \Omega / 2 \pi = 1.50 \) Hz, \( v = 0.033 \) cm\(^2\)/sec, \( F_c = 16.3 \) cm\(^3\)/sec for the instability at the center ring).

Rossby wave, \(-\beta/k^2\), is shown in Fig. 17 for both co- and counter-rotating jets. In addition, Fig. 17 shows \( c/U \) for waves on corotating jets.

The shift in \( c \) for a locked vortex state on a corotating flow is in good accord with that for a sinuous Rossby wave. The agreement with Eq. (5), however, does not distinguish between the fast and slow mode.\(^{22}\) The wave speeds for corotating locked states are all significantly less than \( U/3 \), as shown by the filled squares in Fig. 17, so the system clearly selects the slow mode. del-Castillo-Negrete and Morrison\(^{21,22}\) argue that the slow mode is selected because it has the larger growth rate.

For a counter-rotating jet, the shift in \( c \) from the characteristic velocity is zero within experimental uncertainty. These dispersionless waves are clearly not the sinuous mode predicted by Lipp.\(^{19}\) An explanation may be provided by Howard's semicircle theorem,\(^{2}\) which, for a neutral mode, states that the wave speed falls between the maximum and minimum jet velocities. Agreement with Eq. (5) for the counter-rotating jets would require wave speeds greater than the peak jet velocity. A dispersionless neutral solution was predicted for the counter-rotating Bickley jet [\( U_0 \sim \text{sech}^2(y) \)],\(^{26,51}\) but this mode has \( c = U \), rather than \( c = 2U/3 \), as in the experiments. The locked counter-rotating states are indistinguishable from two dispersionless, \( \beta = 0 \) (Kelvin–Helmholtz type) vortex chains.

Streak photographs of typical locked vortex chains for the narrow forcing configuration are shown in Fig. 18. The vortices are more tightly nested for counter-rotating jets [Fig. 18(b)] than for corotating jets [Fig. 18(a)]. Since rotating states are indistinguishable from two dispersionless, \( \beta = 0 \) (Kelvin–Helmholtz type) vortex chains.

FIG. 16. Velocity power spectra obtained for a corotating jet at the center and outer forcing rings near the transition to locking: (a) and (b), unlocked vortex chains, \( F = 20.0 \) cm\(^3\)/sec; (c) and (d), locked vortex chains, \( F = 22.6 \) cm\(^3\)/sec; (e) and (f), locked vortex chains at the original \( F, 20.0 \) cm\(^3\)/sec (corotating jet, \( \Omega / 2 \pi = 1.50 \) Hz, \( v = 0.033 \) cm\(^2\)/sec, \( F_c = 16.3 \) cm\(^3\)/sec for the instability at the center ring).

FIG. 17. Wave speed and wave speed shifts for locked vortex states: \( \square \), corotating flow; \( \Delta \), counter-rotating flow. The open symbols denote the shift of the wave speed from the characteristic jet velocity (see text), normalized by velocity shift predicted by Eq. (5). The filled squares denote the ratio \( c/U \) of the wave speed to the peak jet velocity for corotating flows. The corotating waves speeds are all significantly less than \( U/3 \) (dashed line) [narrow forcing, \( \Omega / 2 \pi = 1.50 \) Hz, \( v = 0.033 \) cm\(^2\)/sec, \( F_c = 16.3 \) (13.3) cm\(^3\)/sec for the instability at the center ring for the corotating (counter-rotating) jet].
counter-rotating locked states propagate faster than their corotating counterparts, the resonances for the counter-rotating case are closer to the center of the jet, resulting in the tight nesting of the vortex chains.\(^{21}\) The nested vortex structure seen for both co- and counter-rotating locked states could correspond to either a Rossby wave or to a simple merger of two interacting shear-induced chains, as would occur even in the absence of rotation and/or the beta effect; in both cases the streamfunction would have the form shown in Fig. 1(b). The measurements of the wave speeds shown in Fig. 17 distinguish between these possibilities.

V. DISCUSSION

The primary instability of the axisymmetric jet in these experiments has many of the properties of shear-induced instabilities in rotating systems without a beta effect.\(^{6,7,20}\) For flows with moderate Rossby number \(\varepsilon \approx 0.1\), the propagation speeds of the vortex chains near the onset of instability are approximately equal to the average velocity of the shear layer, as for a shear-induced vortex chain. In addition, the critical Reynolds numbers are close in magnitude to the predictions in the absence of beta effects.\(^{7,20}\) However, there is a 20\% difference between the Reynolds numbers for the onset of instability of corotating and counter-rotating jets, a difference that can only be explained by the beta effect. At large rotation rates (small \(\varepsilon\)) the wave speed of a corotating jet at onset of instability falls increasingly below the average jet velocity; this suppression of the wave speed of a corotating jet is a signature of the beta effect (Sec. II C).

The measured forcing levels, mode numbers, and wave speeds of the vortex chains at the onset of the primary instability are in good accord with the analysis of Lee and Marcus\(^{20}\) for both corotating and counter-rotating jets.

With increasing forcing the two vortex chains increase in size and interact strongly. At sufficiently high forcing the inner and outer vortex chains lock together at the same frequency and mode number. The increase in the characteristic length scale (width of the shear layers) as the vortex chains grow and eventually lock reduces the Rossby number by an order of magnitude below typical values near the onset of instability. The Rossby number for the locked states, typically 0.01, is small enough so that Coriolis effects dominate the dynamics. The wave speed for the corotating locked state is significantly smaller than the characteristic jet velocity. This speed is in accord with the prediction for a “slow” sinuous Rossby wave on a sech\(^2\)(\(r\)) jet.\(^{9,22}\) In contrast, counter-rotating locked vortex chains propagate at the characteristic jet velocity, quite different from the speed of a Rossby wave (cf. Fig. 17).

Experiments are currently examining the dynamics in this system at higher Reynolds numbers, emphasizing transport and mixing properties and the transition to spatiotemporal chaos.

ACKNOWLEDGMENTS

We thank D. del Castillo, C. H. Lee, P. S. Marcus, and P. J. Morrison for helpful discussions, and we acknowledge the contributions of S. D. Meyers to the redesign of the tank. We also thank G. Flierl for his assistance in the derivation of Eq. (5).

This work was supported by the Office of Naval Research, Grant No. N00014-89-J-1495.

---

7. M. Niino and N. Misawa, “An experimental and theoretical study of...
Equation (5) can also be derived for the sinuous mode by using the fact that both $U_0$ and $\phi$ have a sech$^2(p)$ dependence for the neutral mode. If $U_0$ and $\phi$ are substituted into (1) and the same powers of sech$(ly)$ are equated, resulting in (5).

The reverse is true if $\beta$ is negative, i.e., Rossby waves propagate in the direction of the rotation. Throughout the rest of the paper, $\beta$ will be assumed to be positive, unless otherwise stated.


Manin and Chernous'ko (Ref. 18) compared some aspects of instability with a Rayleigh-Kuo analysis. They did not measure the wave speeds.

Technically, the profile is a dealised delta function constructed from a series of 65 Chebyshev polynomials.

For larger rotation rates, thermal convection becomes difficult to control, since centrifugal forces grow as $\Omega^2$.


Equation (3) can also be derived for the sinuous mode by using the fact that both $U_0$ and $\phi$ have a sech$^2(p)$ dependence for the neutral mode. If $U_0$ and $\phi$ are substituted into (1) and the same powers of sech$(ly)$ are equated, $l$ can be expressed as a function of $k$, $U_0$, and $c$, and can be eliminated, resulting in (5).


The reverse is true if $\beta$ is negative, i.e., Rossby waves propagate in the direction of the rotation. Throughout the rest of the paper, $\beta$ will be assumed to be positive, unless otherwise stated.